On the algebraicization of certain Stein manifolds.

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Abstract. To every real analytic Riemannian manifold M there is associated a complex structure on a neighborhood of the zero section in the real tangent bundle of M. This structure can be uniquely specified in several ways, and is referred to as a Grauert tube. We say that a Grauert tube is entire if the complex structure can be extended to the entire tangent bundle. We prove here that the complex manifold given by an entire Grauert tube is, in a canonical way, an affine algebraic variety. In the special case $M = S^2$, we show that any entire Grauert tube associated to a metric (not necessarily round) on M must be algebraically biholomorphic to the Grauert tube of the round metric, that is, the non-singular quadric surface in $\mathbb{C}^3$. (This second result has been discovered independently by Totaro.)

1. Introduction

By a *Grauert tube* we mean a smooth complex manifold X of complex dimension n endowed with a C^5 non-negative strictly plurisubharmonic exhaustion function $\tau\colon X \to [0, R)$ whose square root is plurisubharmonic and satisfies the complex homogeneous Monge-Ampère equation,

$$\partial\bar{\partial}(\sqrt{\tau})^n = 0, \tag{1.1}$$

on $X \setminus \{\tau = 0\}$. That τ is an exhaustion means that for all $0 \le r < R$, $\{x \in X \mid 0 \le \tau(x) \le r^2\}$ is compact. If the maximal radius $R = \infty$ we call the tube *unbounded* or *entire*. The simplest example of such is the affine 2-quadric

$$\mathcal{Q} = \{(z_1, z_2, z_3) \in \mathbf{C}^3 \mid z_1^2 + z_2^2 + z_3^2 = 1\} \subset \mathbf{C}^3.$$

with

$$\sqrt{\tau} = \frac{1}{2}\cosh^{-1}(\|z\|^2).$$

By a classical theorem of Remmert, since τ is strictly plurisubharmonic and X is Stein, X can be properly holomorphically embedded in $\mathbf{C}^{\mathbf{N}}$ for $N >> 1$. The main result in this paper is that an unbounded Monge-Ampère exhaustion ensures the existence of an *algebraic* embedding in $\mathbf{C}^N$, similar to the simple example $\mathcal{Q}$ above. We show,

Theorem 1. *Any entire Grauert tube is an affine algebraic manifold.*

The functions giving the algebraic structure to X can be described as the field of quotients of holomorphic functions f on X satisfying

$$\limsup_{r\to\infty} \frac{1}{r} \int_{\{f(\tau)=r\}} \log_+ \mid f \mid \tau^{-\frac{n}{2}}\, d\tau \wedge (\sqrt{-1}\partial\bar{\partial}\tau)^{n-1} < \infty.$$

To place this in context, recall the characterization of $\mathbf{C}^n$ by a "parabolic exhaustion" τ in [**16**], [**4**] where a logarithm replaces the square root in (1.1). In [**4**] the second named author conjectured that manifolds with special exhaustion functions satisfying a Monge-Ampère equation were affine

algebraic. Related to this conjecture is the characterization of affine algebraic manifolds by exhaustions with certain growth conditions given by J.-P. Demailly in his monograph [**7**]. In fact, our task in proving Theorem 1 will be to show that a pair of real valued functions with the growth properties as in Demailly's theorem exists in any entire Grauert tube. The holomorphic functions on X mentioned above will have finite "degree" in the sense of Demailly's result.

To define that pair of real valued functions and to find bounds for their growth we make use of the foliation associated to the function τ, the "Monge-Ampère foliation". Properties of this foliation for parabolic τ were treated in [**4**] and by P.-M. Wong in [**20**], but unlike that case, where the singular set $\{\tau = 0\}$ of the foliation is just a point, in our present situation the singularity set $M := \{\tau = 0\}$ is a totally real and real analytic n-dimensional submanifold. In light of the examples by G. Patrizio and P.-M. Wong in [**14**], this situation was further analyzed by L. Lempert and R. Szőke in [**12**] and independently by V. Guillemin and M. Stenzel in [**9**]. Both in [**12**] and [**9**] it is shown that a Grauert tube has as a model the (co)tangent bundle of M endowed with a certain "adapted complex structure" canonically induced from the Riemannian metric that is inherited by M from the Kähler metric on X with potential τ. However some aspects of the foliation emphasized in [**12**] will be the most useful for our purposes, for instance, the "holomorphic Jacobi fields", that give outside a discrete set a trivialization of the holomorphic tangent bundle along a leaf. One of the functions to be estimated in our proof is a Ricci curvature and thus we are led to work with holomorphic $(n, 0)$ forms on X. Using real analyticity of the foliation off the singular set together with the Bott connection we show that the Riemannian volume form on M extends to a holomorphic "volume form" giving a global trivialization of the canonical bundle of X. In the process we introduce certain "holomorphic Fermi fields" along the holomorphic parametrizations of the leaves of the foliation. Then, the estimates for the growth of that Ricci curvature will essentially follow from estimating the growth of the pairing of the Fermi and the holomorphic Jacobi fields, leafwise, while the compactness of the level sets of τ show that such bounds are uniform.

It follows directly from the proof above that the action of the isometry group of the Riemannian manifold M complexifies naturally to an action on X by *algebraic transformations*. See [**18**]. The complex manifold X also admits an anti-holomorphic involution whose fixed points are the submanifold M. The algebraic embedding of Theorem 1 can be taken to map this involution onto complex conjugation in $\mathbf{C}^{\mathbf{N}}$, and M appears as the real points of the algebraic variety X. Similar to the argument outlined above, one shows that the metric tensor on M has a holomorphic extension to all of X, and in section 6, Theorem 3, we show that this holomorphic form is algebraic, as well. Thus, the metric which gives rise to the entire Grauert tube must necessarily be algebraic.

There are interesting open questions concerning existence and uniqueness of entire Grauert tubes, and we will comment on them in the last section. For now, we state the following result which we prove in this paper as an application of Theorem 1 and results of L. Lempert and R. Szőke [**12**].

Theorem 2. *Let X be an unbounded Grauert tube of complex dimension 2. Then, denoting by $\simeq$ biholomorphic equivalence, either*

$$X \simeq \begin{cases} \mathbf{C}^* \times \mathbf{C}^* \ \ (\ \textit{or}\ (\mathbf{C}^* \times \mathbf{C}^*)/\mathbf{Z_2}) \\ \mathcal{Q} \ \ (\ \textit{or}\ \mathcal{Q}/\mathbf{Z_2}) \end{cases} .$$

Here $\mathbf{C}^*$ is the punctured complex plane and $\mathcal{Q}$ is the affine 2-quadric described above.

A version of this theorem is due independently to B. Totaro [**19**]. We thank Professor Totaro for making his preprint available to us.

2. Holomorphic extension of the Riemannian volume form

We now recall some basic facts necessary to get started (cf. [**12**], [**9**], [**5**]). Let (X, τ) be a Grauert tube of complex dimension n. The Monge-Ampère equation implies that $d\tau \equiv 0$ precisely at M, the zero set of τ. In fact M is a totally real manifold of dimension n, the fixed-point set of an anti-holomorphic involution σ, and X is diffeomorphic to the tangent bundle of M. The tube has a canonical foliation, singular along M, the leaves of which are Riemann surfaces whose tangential directions correspond to the kernel of $\partial\bar{\partial}\sqrt{\tau}$. The Kähler metric on $\mathcal{X}$ with fundamental form $\sqrt{-1}\partial\bar{\partial}\tau$ provides M with a Riemannian metric g. By the regularity theorem of L. Lempert in [**11**] τ is real analytic and so is g . Now, one can reconstruct the tube as the tangent bundle of M endowed with a complex structure $\mathbf{J}$ canonically induced from the Riemannian metric g (and hence "adapted" to g). In that model, each leaf intersects M along a geodesic of g.

The way in which the Riemannian geometry of M manifests itself via holomorphic objects is crucial in the understanding of a Grauert tube. The following is at the heart of our argument.

Proposition 2.1. *The $(n, 0)$ part of the Riemannian volume form of (M, g) can be extended as a non-vanishing holomorphic form $\mathcal{V}$ to all of X.*

Proof. In local real analytic coordinates $x_1, \cdots, x_n$ valid on a ball $\mathcal{U} \subset M$ the Riemannian volume form has the expression $\sqrt{\det \tilde{g}}\, dx_1 \wedge \cdots \wedge dx_n$ where the determinant of the matrix $\tilde{g}_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ is non-vanishing and real analytic. Thus, $\big(\sqrt{\det \tilde{g}}\, dx_1 \wedge \cdots \wedge dx_n\big)^{n,0}$ has a non-vanishing holomorphic extension to a certain neighborhood in X containing $\mathcal{U}$. By compactness of M and uniqueness of analytic continuation, the volume form on M can be extended as a non-vanishing holomorphic form $\mathcal{V}$ beyond the level set $\{\tau = \epsilon\}$ for $\epsilon > 0$ small enough.

We are first going to extend this holomorphic volume form as a non-vanishing form leaf by leaf to all of X and then use real analyticity of the foliation to show that the result is a holomorphic form.

Fix a leaf $\mathcal{L}_\gamma$, intersecting M along a geodesic γ. Choose a parametrization of γ by arc length and let

$$P_\gamma(x+\sqrt{-1}y) = y\dot{\gamma}(x) \tag{2.1}$$

be the holomorphic map parametrizing $\mathcal{L}_\gamma$.

2.1. Holomorphic Fermi Fields. We will define a special holomorphic frame along P_γ trivializing the holomorphic vector bundle $P_\gamma^*(T^{1,0}(X))$. To do this, we need to recall the construction of the so-called Fermi coordinates. Consider a parallel orthonormal frame along γ $\{E_1(s), \cdots, E_n(s)\}$ so that $E_n(s) = \dot{\gamma}(s)$, for s in a neighborhood of s^0 in $\mathbf{R}$. One then defines the Fermi coordinates s_i, whose domain is some neighborhood $\mathcal{U}$ of $\gamma(s^0)$ in M, by the formulas

$$s_i\big(\exp^N_{\gamma(s)}(\sum_k^{n-1} a_k E_k(s))\big) = \begin{cases} a_i \textit{ for } 1 \le i \le n-1 \\ s \textit{ for } i = n \quad , \end{cases} \tag{2.2}$$

where $\exp^N$ is the exponential map restricted to the normal bundle of γ.

Let z_i, $1 \le i \le n$, be the holomorphic extension of the real analytic function s_i to a neighborhood $\mathcal{W}$ in X containing $\mathcal{U}$ and let

$$Z_i := \frac{\partial}{\partial z_i}, \quad 1 \le i \le n,$$

be the corresponding holomorphic vector fields. The expression

$$F_i := Z_i \circ P_\gamma$$

defines a holomorphic frame trivializing $P_\gamma^*(T^{1,0}(\mathcal{X})) \mid_{P_\gamma^{-1}\mathcal{W}}$ and by iterated use of the map P_γ we get a holomorphic trivialization of $P_\gamma^*(T^{1,0}(X)) \mid_{\mathcal{N}}$ on a neighborhood $\mathcal{N}$ of the real axis in $\mathbf{C}$. (Here P_γ "unwinds" the geodesic so that the parallel translation of a frame E_i, and hence the Z_i's, correspond to well-defined sections.)

The next step is to show that the F_i's can be extended from $\mathcal{N}$ to all of $\mathbf{C}$ and that their extensions trivialize $P_\gamma^*(T^{1,0}(X))$. Note that F_n does extend as a non-vanishing holomorphic section, since from (2.1), (2.2) and $E_{n(s)} = \dot{\gamma}(s)$ we have the identification, for all ω in $\mathcal{N}$,

$$F_n \circ P_\gamma = \frac{\partial}{\partial \omega}, \tag{2.3}$$

thus defining F_n throughout $\mathbf{C}$.

On $\mathbf{C}$ we have the short exact sequence of vector bundles

$$0 \to T^{(1,0)}(\mathbf{C}) \to \mathbf{P}_\gamma^*(\mathbf{T^{1,0}}(\mathbf{X})) \to \mathbf{N} \to \mathbf{0},$$

where N is the normal bundle of $\mathcal{L}_\gamma$ pulled back to $\mathbf{C}$. $T^{(1,0)}(\mathbf{C})$ is spanned by $Z_n = \frac{\partial}{\partial \omega}$, and has a connection which makes Z_n covariant constant.

Similarly, N has a holomorphic connection, the Bott connection, defined locally, for Z a holomorphic section of N, by

$$\nabla_{Z_n} Z = [Z_n, \tilde{Z}] \bmod T^{(1,0)}(\mathbf{C}),$$

where $\tilde{Z}$ is a lift of Z to a holomorphic vector field on X in a neighborhood. It is well known that this is independent of the lift $\tilde{Z}$. We want to glue these two connections together to get a holomorphic connection on $P_\gamma^*(T^{1,0}(X))$. To do this, it suffices to show that the short exact sequence splits holomorphically in a natural way. Indeed, off the real axis, the one form ∂u is holomorphic and its vanishing defines a complementary bundle $E :=$ Ker ∂u to $T^{(1,0)}(\mathbf{C})$, which enables us to define a connection on $P_\gamma^*(T^{1,0}(X))$ as the sum of the two pieces already discussed on the two summands (lifting the Bott connection to the complement of $T^{(1,0)}(\mathbf{C})$). We need to show that this splitting of the bundle $P_\gamma^*(T^{1,0}(X))$ can be continued holomorphically across the real axis.

To do this, we note that since $u = \sqrt{\tau}$, we have that $\partial u = \frac{1}{2u}\partial\tau$, away from M. On the upper half plane in $\mathbf{C}$, we have that $u = y_n = Im\ \omega$; on the lower half plane, $u = -y_n = -Im\ \omega$. Let us check that each of the functions $-2\sqrt{-1} < \partial u, Z_j >$ on the upper half plane near the real axis extends holomorphically across the real axis. More precisely, we wish to show they extend holomorphically past the real axis and

$$-2\sqrt{-1} < \partial u, Z_j >= \delta_{j,n}$$

along the axis, where $\delta_{j,n}$ is the Kronecker δ. Note first that for $j = n$, the conclusion is obvious, since $< \partial u, Z_n > \equiv -\frac{\sqrt{-1}}{2}$. For the others, we first note that

$$< \partial u, Z_j >= \frac{1}{2u} Z_j(\tau) = \frac{1}{2y_n}\frac{\partial\tau}{\partial z_j}.$$

Since τ has a minimum along M, $\frac{\partial\tau}{\partial z_j} \equiv 0$ along γ (or along the real axis in $\mathbf{C}$). Restricted to $\mathcal{L}_\gamma$ (pulled back to $\mathbf{C}$) where $y_n = 0$ is a real analytic defining equation for γ (the real axis), we conclude that y_n divides $\frac{\partial\tau}{\partial z_j}$ real analytically in a neighborhood of γ. In particular, the extended function has value along γ given by $\frac{1}{2}\frac{\partial^2\tau}{\partial y_n \partial z_j}$. But since $\frac{\partial\tau}{\partial z_j} \equiv 0$ along γ, we have $\frac{\partial^2\tau}{\partial x_n \partial z_j} \equiv 0$ along γ, and hence

$$\frac{\partial^2\tau}{\partial y_n \partial z_j} = -2\sqrt{-1}\frac{\partial^2\tau}{\partial \bar{z}_n \partial z_j},$$

along γ. Finally, $\frac{\partial^2\tau}{\partial \bar{z}_n \partial z_j} \equiv 0$ along γ because the τ-metric restrictsto give our original metric on M and because of the orthogonality properties of the Fermi fields along γ.

Note that by Schwarz reflection, the extension of $-2\sqrt{-1}\partial u$ to the lower half plane agrees with $2\sqrt{-1}\partial u$ there. Thus, we have extended the holomorphic complement to $T^{(1,0)}(\mathbf{C})$ holomorphically across the real axis, and therefore constructed a holomorphic connection on the bundle $P_\gamma^*(T^{1,0}(X))$ over all of $\mathbf{C}$.

On $\mathcal{N} \cap \gamma$, the F_i's lie in the subbundle E complementary to $T^{(1,0)}(\mathbf{C})$. Since they are holomorphic sections, they must lie in E on all of $\mathcal{N}$. Then the F_i's can be taken to be the lifts to $P_\gamma^*(T^{1,0}(X))$ of the sections of N on $\mathcal{N}$ given by the Fermi coordinate fields $Z_1, \cdots, Z_{n-1}$, and these F_i are covariant constant since they are coordinate fields. Finally, this allows us to extend them so that together with F_n they form a holomorphic frame trivializing $P_\gamma^*(T^{1,0}(X))$.

To complete the proof of Proposition 2.1, let us consider the dual forms of the F_i, i.e., the sections F_i^* of the holomorphic bundle $P_\gamma^*(T^{1,0}(X))^*$. Since along $\mathcal{N}$

$$F_1^* \wedge \cdots \wedge F_n^* = \mathcal{V} \circ P_\gamma,$$

we can now extend $\mathcal{V}$ from $\mathcal{W} \cap \mathcal{L}_\gamma$ to all of $\mathcal{L}_\gamma$.

Because $X \setminus M$ is foliated by the $\mathcal{L}_\gamma$'s, the construction above performed for each geodesic γ defines a form $\mathcal{V}$ for every point of X. In principle we only know this form to be holomorphic in the region $\{0 \leq \tau < \epsilon\}$, but because of real analyticity of τ we have the following

Claim: The leaf-wise extension $\mathcal{V}$ is holomorphic on X.

To see this, it suffices to observe that the $(n, 0)$-form $\mathcal{V}$ is real analytic on X, since we already know that it is holomorphic in a neighborhood of M, and would be holomorphic on all of X by analytic continuation. On the other hand, the geodesic γ (and consequently the leaf $\mathcal{L}_\gamma$) depends real analytically on $2n-2$ real parameters $s = (s_1, \cdots, s_{2n-2})$, as do the associated Fermi coordinates $x = x(s)$ and their complexifications $z = z(s)$, and the corresponding Fermi fields $Z_i(s)$, locally. Finally, since the Monge-Ampère solution u is $\mathcal{C}^\omega$ in all arguments off M, the complementary bundles $E(s)$ also vary real analytically. The extensions of the $Z_i(s)$ along the leaves are achieved by solving systems of holomorphic ordinary differential equations which depend real analytically on the parameters s, and with real analytically varying initial conditions, hence they depend real analytically on all parameters. This concludes the proof of Proposition 2.1.

□

3. J.-P. Demailly's characterization of affine varieties

Let's recall the following very pretty result, Theorem 9.1′ in [**7**], stated here in less generality than the original version, just to the degree needed for our purposes.

Theorem: (Demailly [**7**]) *Let X a complex manifold of complex dimension n with finite dimensional cohomology groups. Assume that there is a smooth strictly plurisubharmonic exhaustion function ϕ on X with*

$$\int_X (\sqrt{-1}\partial\bar{\partial}\phi)^n < \infty.$$

Let $R(\beta)$ be the Ricci form of the Kähler metric induced by the potential e^ϕ and put $\beta = \sqrt{-1}\partial\bar{\partial}e^\phi$. If there are non-negative constants c and c', and a

C^2 function ψ so that

$$R(\beta)+\sqrt{-1}\partial\bar{\partial}\psi \geq 0, \quad \textit{and} \quad \int_X e^{c\psi-c'\phi}\beta^n < \infty$$

then X is biholomorphic to an affine algebraic variety.

The algebraic embedding of X into $\mathbf{C}^N$, $N >> 1$, is via holomorphic functions in the algebra A_ϕ consisting of "ϕ-polynomials", holomorphic functions f in X such that their "degree"

$$\limsup_{r\to\infty} \frac{1}{r}\int_{\phi=r} \log_+ \mid f \mid d\phi \wedge (\partial\bar{\partial}\phi)^{n-1} < \infty.$$

In this section we propose for a given Grauert tube $\big(X,\tau\big)$ of infinite radius functions ϕ and ψ that will meet the conditions in Demailly's theorem. The proof that ϕ indeed satisfies the required growth conditions occupies the next two sections.

The ϕ will be a function solely of τ and the ψ will be defined in terms of the holomorphic extension of the Riemannian volume form from Proposition 2.1. The choices made are not unique but are in fact quite natural (cf. Remark 3.1).

Proposition 3.1. *Let X be an unbounded tube with Monge-Ampère exhaustion function τ. Then*

$$\phi = \ln(1+\cosh\sqrt{\tau}) \tag{3.1}$$

is strictly plurisubharmonic and $\int_X(\sqrt{-1}\partial\bar{\partial}\phi)^n < \infty$.

Proof. We have $\phi = f(\tau)$ and $\partial\bar{\partial}\phi = f''(\tau)\partial\tau\wedge\bar{\partial}\tau + f'(\tau)\partial\bar{\partial}\tau$ where

$$f'(\tau) = \frac{1}{1+\cosh\sqrt{\tau}}\frac{\sinh\sqrt{\tau}}{2\sqrt{\tau}}, \quad f''(\tau) = \frac{1}{4\sqrt{\tau}(1+\cosh\sqrt{\tau})}\Big(1-\frac{\sinh\sqrt{\tau}}{\sqrt{\tau}}\Big).$$

Strict plurisubharmonicity easily follows from the inequalities $f'(\tau) > 0$ and $2\tau f''(\tau) + f'(\tau) > 0$ and the splitting discussed in the proof of Proposition 2.1, $T^{1,0}(X)|_{X\setminus M} = \mathbf{C}\xi \oplus$ *Kernel* $\partial\tau$, with ξ in the tangential direction to each leaf.

On the other hand, from the Monge-Ampère equation we get

$$\partial\tau\wedge\bar{\partial}\tau\wedge(\partial\bar{\partial}\tau)^{n-1} = \frac{2\tau}{n}(\partial\bar{\partial}\tau)^n \tag{3.2}$$

and by letting

$$\delta_0(\tau) := \frac{1}{2^n}\Big(\frac{\sinh\sqrt{\tau}}{\sqrt{\tau}}\Big)^{n-1},$$

we have

$$(\partial\bar{\partial}\phi)^n = \frac{\delta_0(\tau)}{(1+\cosh\sqrt{\tau})^n}(\partial\bar{\partial}\tau)^n.$$

Putting $X' := \{1 \leq \tau < \infty\}$ it follows that there is a $C > 0$ so that

$$\int_X(\sqrt{-1}\partial\bar{\partial}\phi)^n \leq C\int_{X'} e^{-\sqrt{\tau}}(\sqrt{-1}\partial\bar{\partial}\tau)^n. \tag{3.3}$$

But under the identification of X with the tangent bundle of $M = \{\tau = 0\}$ endowed with the Riemannian metric g, τ corresponds to the g-length squared in $T\mathcal{M}$ and $\sqrt{-1}\partial\bar{\partial}\tau$ to the symplectic 2-form induced via g by the tautological 2-form in the cotangent bundle T^*M. Thus, we have

$$\begin{aligned}\int_{X'} e^{-\sqrt{\tau}}(\sqrt{-1}\partial\bar{\partial}\tau)^n =& n\int_{X'} \frac{e^{-\sqrt{\tau}}}{2\tau}\partial\tau\wedge\bar{\partial}\tau\wedge(\sqrt{-1}\partial\bar{\partial}\tau)^{n-1}\\ =& C' \ \textit{Volume of}\ (M,g)\int_1^\infty \lambda^{n-2}e^{-\sqrt{\lambda}}d\lambda \leq \infty\end{aligned}$$

for a constant C' depending on n, showing that $\int_X(\sqrt{-1}\partial\bar{\partial}\Phi)^n$ is finite. $\square$

Next, in order to find a candidate for ψ, we consider the Ricci form $R(\beta)$ of the Kähler metric β induced by the potential

$$e^{\phi} = 1 + \cosh\sqrt{\tau}. \tag{3.4}$$

We have,

$$\partial\bar{\partial}(1+\cosh\sqrt{\tau}) = \frac{\sinh\sqrt{\tau}}{2\sqrt{\tau}}\Big(\frac{1}{2\tau}(\frac{\sqrt{\tau}}{\tanh\sqrt{\tau}}-1)\partial\tau\wedge\bar{\partial}\tau + \partial\bar{\partial}\tau\Big), \tag{3.5}$$

and hence, using (3.2),

$$(\partial\bar{\partial}(1+\cosh\sqrt{\tau}))^n = \delta_0 \cosh\sqrt{\tau}\ (\partial\bar{\partial}\tau)^n. \tag{3.6}$$

Let $\mathcal{V}$ be the non-vanishing holomorphic $(n,0)$-form on X from Proposition 2.1 extending the Riemannian volume form on M, and let δ be the function

$$\delta\colon X \to \mathbf{R} > 0$$

globally defined by

$$\mathcal{V}\wedge\bar{\mathcal{V}} = \frac{(-1)^{\frac{n^2}{2}}}{n!}\delta(\sqrt{-1}\partial\bar{\partial}\tau)^n. \tag{3.7}$$

By (3.6) and (3.7), the Ricci form $R(\beta)$ has globally the expression

$$R(\beta) = \frac{1}{\sqrt{-1}}\partial\bar{\partial}\ln\big(\frac{\delta_0}{\delta}\ \cosh\sqrt{\tau}\big),$$

and if we simply let

$$\psi := \ln\big(\frac{\delta_0}{\delta}\ \cosh\sqrt{\tau}\big), \tag{3.8}$$

then

$$R(\beta) + \sqrt{-1}\partial\bar{\partial}\psi = 0. \tag{3.9}$$

Consider the following

PROPOSITION 3.2. *$\exists$ constants $C > 0$ and $k > 0$ so that*

$$1/\delta \leq Ce^{k\sqrt{\tau}}. \tag{3.10}$$

Since clearly we can find a constant $c > 0$ so that $\cosh\sqrt{\tau}\ \delta_0(\tau) < ce^{(n+1)\sqrt{\tau}}$, then Proposition 3.2 has the following (cf. (3.1), (3.8))

Corollary 3.2.1. *∃ constants $A > 0$ and $B > 0$ so that*

$$\psi \leq A\phi + B. \tag{3.11}$$

3.0.1. *Proof of Theorem 1.*

Proof. We prove Proposition 3.2, and hence (3.11), in the next two sections. Assuming those results, by (3.9) and (3.11) the functions ϕ and ψ meet the conditions of Demailly's theorem 9.1' in [**7**].

Indeed, given (3.11) it is immediate to check that there is a constant $c'' > 0$ so that $\int_X e^{\psi - c'\phi}\beta^n$ is bounded above by

$$c'' \int_X e^{(n+1)\sqrt{\tau}}\big(1 + \cosh\sqrt{\tau}\big)^{A-c'}(\sqrt{-1}\partial\bar{\partial}\tau)^n,$$

a finite number for $c' >> A$.

The finiteness of the cohomology groups is satisfied since X is diffeomorphic to the tangent bundle of the compact manifold $\{\tau = 0\}$. □

Remark 3.1. *The choices of ϕ and ψ made above, although not unique, are natural (ϕ was already noticed to have the "finite volume" property in* [**14**]*).*

For example, consider the affine quadric $\mathcal{Q}_n = \{\sum_{i=1}^{n+1}\omega_i^2 = 1\} \subset \mathbf{C}^{n+1} \subset \mathbf{CP}^{n+1}$. By [**14**], $\sqrt{\tau} = \cosh^{-1}\sum \mid \omega_i \mid^2$ and hence ϕ is the restriction of the potential for the Study-Fubini metric in $\mathbf{CP}^{n+1}$. A computation will show that $\delta = \delta_0$ and $\psi = \ln \sum \mid \omega_i \mid^2$. Note that in general δ is not a function of τ alone. It is so precisely when the Riemannian metric induced in $\{\tau = 0\}$ is *harmonic* (cf. [**1**]).

We devote the next two sections to the proof of Proposition 3.2, which will complete the proof of Theorem 1.

4. Proof of Proposition 3.2: Part 1, δ along a leaf.

In this section once again we identify as complex manifolds the Grauert tube X with $(TM, \mathbf{J})$, the tangent bundle of $M = \{\tau = 0\}$ endowed with the adapted complex structure $\mathbf{J}$. Each geodesic γ of (M, g) defines a leaf $\mathcal{L}_\gamma$, which, for a chosen parametrization of γ by arc length, is given a holomorphic parametrization $P_\gamma \colon \mathbf{C} \to M$ defined by

$$P_\gamma(\omega) = P_\gamma(x + \sqrt{-1}y) = y\,\dot{\gamma}(x). \tag{4.1}$$

Our goal is to show that there are constants $C > 0$ and $k > 0$ such that $1/\delta < Ce^{k\sqrt{\tau}}$. We will accomplish this in the next section. In the present section we first obtain a convenient expression of $(1/\delta) \circ P_\gamma$ for each parametrized geodesic γ, given in Proposition 4.1. (In fact we will later be able to show the inequality (5.15), which is stronger than needed here.)

Choose an orthonormal frame of $T_{\gamma(0)}M$ and extend it along γ by parallel translation to obtain the frame $E_1(s), \cdots E_n(s)$, which at each s is an

orthonormal basis of $T_{\gamma(s)}M$. Denote by the same symbols the vector fields on the real axis along P_γ.

Consider now the corresponding holomorphic Fermi fields along P_γ as we defined them in the proof of Proposition 2.1,

$$F_1(\omega), \cdots F_n(\omega), \tag{4.2}$$

and recall, for later reference, that for all $-\infty < x < \infty$, $1 \leq i \leq n$,

$$F_i(x) = E_i^{1,0}(x). \tag{4.3}$$

For all $\omega \in \mathbf{C}$, again from our construction in the proof of Proposition 2.1,

$$\mathcal{V} = \Pi_{i=1}^n F_i^*(\omega),$$

where the F_i^*'s are the forms dual to the F_i's and where, to simplify notation, we also denote by $\mathcal{V}$ the holomorphic section of $P_\gamma^*\big((T^{n,0}(X)^*\big)$ determined by the holomorphic volume form $\mathcal{V}$. Thus

$$\Pi_{i=1}^n F_i^* \wedge \bar{F}_i^* = (-1)^{\frac{n(n-1)}{2}} \mathcal{V} \wedge \bar{\mathcal{V}},$$

i.e.,

$$\Pi_{i=1}^n F_i^* \wedge \bar{F}_i^* = \frac{\delta}{n!}(\partial\bar{\partial}\tau)^n. \tag{4.4}$$

To take advantage of equation (4.4) we will resort to two more sets of vector fields holomorphic along P_γ. The basic reference here is L. Lempert and R. Szőke's work [**12**], where they introduced these vector fields to describe the adapted complex structure.

Let $\xi_i^0(s)$ and $\eta_i^0(s)$, $1 \leq i \leq n$, be the Jacobi fields along γ satisfying "Lagrangian" initial conditions such as

$$\eta_i^0(0) = \nabla_{\dot\gamma(0)}\xi_i^0(0) = 0, \quad \xi_i^0(0) = \nabla_{\dot\gamma(0)}\eta_i^0(0) = E_i(0), \tag{4.5}$$

where ∇ is the Levi-Civita connection on (M, g). Recall that these vector fields are solutions of the differential equation

$$\nabla^2_{\dot\gamma(x)}\xi_i^0(x) + R_\gamma(x)\xi_i^0(x) = 0, \tag{4.6}$$

(and similarly for the η_i's), where $R_\gamma(x) \colon T_{\gamma(x)}M \to T_{\gamma(x)}M$ is the curvature operator along γ given by $v \mapsto R(v, \dot\gamma(0))\dot\gamma$ with R the curvature tensor of (M, g). They determine along P_γ vector fields η_i and ξ_i that are invariant by the geodesic flow and fiberwise rescaling. It is shown in [**12**] that the $(1,0)$ parts of these invariant fields

$$\xi_i^{1,0} = \frac{1}{2}(\xi_i - \sqrt{-1}\mathbf{J}\xi_i), \qquad \eta_i^{1,0} = \frac{1}{2}(\eta_i - \sqrt{-1}\mathbf{J}\eta_i) \tag{4.7}$$

are holomorphic sections of the holomorphic bundle over $\mathbf{C}$ $P_\gamma^*\big(T^{1,0}(X)\big)$ (these are the "holomorphic Jacobi fields" referred to above). Again using the same symbols η_i^0 and ξ_i^0 to denote the induced fields on the real axis along P_γ we have

$$(\eta^0)_i^{1,0}(x) = \eta_i^{1,0}(x), \qquad (\xi^0)_i^{1,0}(x) = \xi_i^{1,0}(x).$$

Now, on $\mathbf{R} \setminus S$, where

$$S := \{x \in \mathbf{R} \mid \xi_1^0(x) \wedge \cdots \wedge \xi_n^0(x) = 0\},$$

a discrete set, for the ξ_i's are solutions along the real axis of the second order differential equations (4.6), we also have

$$\eta_i^0(x) = \sum_{k=1}^{n} h_{ki}^0(x)\xi_k^0(x). \tag{4.8}$$

It follows that along P_γ there are meromorphic functions $h_{ki}(\omega)$ extending $h_{ki}^0(x)$ to $\mathbf{C}$ so that

$$\eta_i^{1,0}(\omega) = \sum_{k=1}^{n} h_{ki}(\omega)\xi_k^{1,0}(\omega), \tag{4.9}$$

which implies

$$\eta_i(\omega) = \sum_{k=1}^{n} \Re h_{ki}(\omega)\ \xi_k(\omega) + \sum_{k=1}^{n} \Im h_{ki}(\omega)\ \mathbf{J}\xi_k(\omega). \tag{4.10}$$

We will need to recall from [**12**] more facts about these meromorphic functions later, but for now we note that the volume form of the Kähler metric given by τ can be expressed as

$$\frac{1}{n!}(\partial\bar{\partial}\tau)^n = \det M\ \Pi_{i=1}^n (\xi_i^{1,0})^* \wedge (\xi_j^{0,1})^*$$

where, again "$*$" denotes dual forms and

$$M_{ij}(\omega) = \partial\bar{\partial}\tau(\xi_i^{1,0}, \xi_j^{0,1}). \tag{4.11}$$

Furthermore, since by the proof of Proposition 2.1 the holomorphic Fermi fields along P_γ are $\mathbf{C}$-linearly independent on all of $\mathbf{C}$ we have

$$\xi_i^{1,0}(\omega) = \sum_{k=1}^{n} A_{ki}(\omega)F_k(\omega), \tag{4.12}$$

where the $A_{ki}(\omega)$ are holomorphic functions on $\mathbf{C}$ so that on $\mathbf{C} \setminus S$

$$\det A(\omega) \neq 0.$$

We thus conclude, noting that the dual forms corresponding to the fields in (4.12) are related by the matrix $(A^{-1})^T$, that

$$\frac{1}{\delta} \circ P_\gamma = \frac{\det M(\omega)}{\mid \det A(\omega) \mid^2} \quad \textit{for all } \Im\omega \neq 0. \tag{4.13}$$

Because τ is an exhaustion function we only need to estimate $1/\delta$ outside a relatively compact neighborhood of M, say on $\tau \geq 1$. In particular, for any geodesic γ, it will suffice to bound $(1/\delta) \circ P_\gamma$ for $y \geq 1$.

We will first deal with an estimate of the numerator of (4.13), our present goal being to express it in terms of the functions $h_{ki}(\omega)$, for which we know how to obtain bounds.

With that in mind, we will rewrite the coefficients $M_{ij}(\omega)$ (4.11) in the form (4.18). Note that at each point $\omega = x + \sqrt{-1}y$ in the leaf with $y > 0$ the span of the vector fields $\{\xi_1, \cdots, \xi_n\}$ and that of the $\{\eta_1, \cdots, \eta_n\}$ are two complementary Lagrangian subspaces of $T_{P_\gamma\omega}\big(TM\big)$ with,

$$\sqrt{-1}\partial\bar{\partial}\tau(\xi_i, \xi_j) = \sqrt{-1}\partial\bar{\partial}\tau(\eta_i, \eta_j) = 0 \tag{4.14}$$

$$\sqrt{-1}\partial\bar{\partial}\tau(\xi_i, \eta_j) = y\delta_{ij}. \tag{4.15}$$

Equations (4.14) and (4.15) are a consequence of the initial conditions (4.5) and the equivariance of the ξ_i, the η_i's and the symplectic form. Indeed, at each point ω in the leaf,

$$K\xi_i(\omega) = \nabla_{P\omega}\xi^0(\pi(P_\gamma\omega)), \quad \pi_*\xi_i(\omega) = \xi_i^0(\pi(P_\gamma\omega)), \tag{4.16}$$

and similarly for the η_i's, where $K\colon T\big(TM\big) \to TM$ is the connection map and $\pi\colon TM \to M$ the natural projection. Moreover, $\sqrt{-1}\partial\bar{\partial}\tau$ which has the expression, always under the tangent bundle identification,

$$\sqrt{-1}\partial\bar{\partial}\tau = g \circ (K \otimes \pi_* - \pi_* \otimes K), \tag{4.17}$$

is invariant by the geodesic flow, and homogeneous of degree one with respect to multiplication along the fibers of TM. Thus, equations (4.14) and (4.15), valid by (4.5) at $P_\gamma^{-1}(\dot{\gamma}(0))$, are so throughout the whole leaf $\mathcal{L}_\gamma$.

It follows that,

$$M_{ij}(\omega) = \frac{1}{2}\sqrt{-1}\partial\bar{\partial}\tau(\xi_i(\omega), \mathbf{J}\xi_j(\omega)),$$

and from (4.10), (4.14) and (4.15), for all ω with $\Im\omega > 0$,

$$M_{ij}(\omega) = \frac{1}{2}\Im\omega(\Im h)^{-1}_{ij}(\omega). \tag{4.18}$$

Now we focus on the denominator of (4.13), again our objective being to express it in terms of the functions $h_{ki}(\omega)$, which we do by obtaining (4.26).

Equation (4.12) reads along the real axis, by (4.3),

$$(\xi_i^0)^{1,0}(x) = \sum_{k=1}^{n} A_{ki}(x)E_k^{1,0}(x). \tag{4.19}$$

But, since $T_{\gamma(x)}M$ is spanned by the $E_k(x)$'s while along the zero section $s\colon M \to TM$ we have $T\big(TM\big)|_M = s_*(TM) \oplus \mathbf{J}s_*(TM)$, it follows that for $-\infty < x < \infty$

$$\xi_i^0(x) = \sum_{k=1}^{n} A_{ki}(x)E_k(x). \tag{4.20}$$

The same arguments concerning the ξ_i's leading to (4.12), (4.19) and (4.20) apply to the η_i's so that for ω in $\mathbf{C}$

$$\eta_i^{1,0}(\omega) = \sum_{k=1}^{n} B_{ki}(\omega)F_k(\omega) \tag{4.21}$$

where B_{ki} are holomorphic on $\mathbf{C}$ and moreover along the real axis

$$\eta_i^0(x) = \sum_{k=1}^{n} B_{ki}(x) E_k(x). \tag{4.22}$$

Of course, from (4.8), (4.12) and (4.21), for all ω outside the discrete subset $S \subset \mathbf{R}$ we have

$$B_{ki}(\omega) = \sum_{s=1}^{n} A_{ks}(\omega) h_{si}(\omega), \tag{4.23}$$

which gives, upon restriction to $\mathbf{R} \setminus S$, double differentiation and use of the Jacobi equations (4.6) together with the fact that the E_i's are parallel along γ, the differential equation (cf. Proposition 6.11 in [**12**]), now in matrix notation,

$$2A'(x){h^0}'(x) + A(x){h^0}''(x) = 0. \tag{4.24}$$

On the other hand, from (4.14), (4.16) and (4.17) it follows that (cf. Proposition 6.10 in [**12**]) $\big(A(x)^T A(x) - A(x)A(x)^T\big)' = 0$. Using this back in (4.24) and taking into account the initial conditions for the ξ_i^0's, and η_i^0's (4.5) for the equations (4.6) which imply $A(0) = {h^0}'(0) = \mathbf{I}$, gives for all ω in $\mathbf{C} \setminus S$,

$$A^T(\omega)A(\omega) = A(\omega)A^T(\omega) = (h'(\omega))^{-1}. \tag{4.25}$$

In particular,

$$\big(\det A(\omega)\big)^{-2} = \det h'(\omega). \tag{4.26}$$

To summarize, in virtue of (4.13), (4.18) and (4.26), we have shown

Proposition 4.1. *Let γ be an arc length parametrized geodesic, P_γ the map (4.1) and h_γ the matrix with components given by (4.9) corresponding to γ. Then for all $y = \Im\omega > 0$,*

$$\frac{1}{\delta(P_\gamma\,\omega)} = y^n \frac{|\det h'_\gamma(\omega)|}{\det(\Im h_\gamma)(\omega)}. \tag{4.27}$$

5. Proof of Proposition 3.2: Part 2, growth estimate for δ.

In this section we will use (4.27) to estimate the growth of δ in terms of τ. This will complete the proof of Proposition 3.2.

We show our bounds along the same lines of R. Szőke's work in [**18**]. Consider the upper half-plane

$$\mathbf{C}^+ = \{\omega = x + \sqrt{-1}y \in \mathbf{C}, x, y \in \mathbf{R} \mid y > 0\}$$

and the space of symmetric complex valued $n \times n$ matrices with positive semi-definite imaginary part,

$$\mathcal{H}^n = \{M \in M^n_{\mathbf{C}} \mid M^T = M, \Im M > 0\}.$$

PROPOSITION 5.1. *Let $F\colon K \times \mathbf{C}^+ \to \mathcal{H}^{\mathbf{n}}$ be a continuous map, K compact, such that for each point u in K, its restriction to $\{u\} \times \mathbf{C}^+$ is holomorphic. Then for all (u, ω) in $K \times U_1$ where*

$$U_1 = \{\omega \in \mathbf{C} \mid -1 < x < 1, y \geq 1\}, \tag{5.1}$$

there are constants $C_1 > 0$ and C_2 such that

$$\det \Im F(u, \omega) \geq C_1 y^{-n}, \tag{5.2}$$

and, putting $F' = \frac{\partial F}{\partial \omega}$,

$$\mid \det F'(u, \omega) \mid \leq C_2. \tag{5.3}$$

PROOF. The inequality (5.2) would follow from Lemma 8.4 in [**18**], but we include a proof here (along the lines of Szőke's) for reference and because most of its ingredients are needed to show inequality (5.3).

Inequality (5.2) follows from the compactness of K and the continuity of F simply by letting

$$C_1 = \min_{u \in K} \det \Im F(u, \sqrt{-1}),$$

since for fixed u in K, along $\{u\} \times U_1$

$$\det \Im F(u, \omega) \geq (4y)^{-n} \det \Im F(u, \sqrt{-1}). \tag{5.4}$$

To show (5.4), recall from [**12**] that for fixed u in K, the assumptions on F imply that we have the Fatou representation

$$\Im F(u, \omega) = y\, \alpha(u) + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\mu(u, t)}{(x - t)^2 + y^2} \tag{5.5}$$

for $\alpha(u)$ a non-negative symmetric constant matrix and $d\mu(u, t)$ a symmetric non-negative matrix of signed Borel measures on the real line, i.e., for all points $(v_1, \cdots, v_n)$ in $\mathbf{R}^n$

$$\sum_{ik} d\mu_{ik}(u, t) v_i v_k \geq 0 \tag{5.6}$$

as a measure, where for all $1 \leq i, j \leq n$ $\int \frac{|d\mu_{ij}(u,t)|}{1+t^2} < \infty$.

Now , for $|x| < 1$ and $y \geq 1$ we have that for all t in $\mathbf{R}$, $\frac{1}{1+t^2} < \frac{4y^2}{(x-t)^2+y^2}$ and thus, in virtue of (5.6) and by the identity obtained from letting $y = 1$ in (5.5), we get

$$\Im F(u, \omega) \geq y\, \alpha(u) + \frac{1}{4y\pi} \int_{-\infty}^{\infty} \frac{d\mu(u, t)}{t^2 + 1} = \frac{(y^2 - 1)}{4y} + \frac{1}{4y} \Im F(u, \sqrt{-1})$$

and so,

$$\Im F(u, \omega) \geq \frac{1}{4y} \Im F(u, \sqrt{-1}). \tag{5.7}$$

Finally, take determinants and recall that $M_1 \geq M_2$ for M_i non-negative real symmetric matrices implies $\det M_1 \geq \det M_2$. Thus, inequality (5.4), and hence (5.2), are proved.

We now show inequality (5.3). By differentiating F with respect to ω, using (5.5) and that F is holomorphic in ω,

$$F'(u,\omega) = \alpha(u) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\mu(u,t)}{(\omega - t)^2} \tag{5.8}$$

and thus, for all $1 \leq i, j \leq n$,

$$\mid F'_{ij}(u,\omega) \mid \leq \mid \alpha_{ij}(u) \mid + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mid d\mu_{ij}(u,t) \mid}{\big((x-t)^2 + y^2\big)^2}. \tag{5.9}$$

It is not hard to find appropriate bounds for the terms in the right-hand side above. Indeed, since $\alpha(u)$ is a non-negative symmetric matrix, then for all $1 \leq i, j \leq n$

$$\alpha_{ii}(u) \geq 0, \qquad \mid \alpha_{ij}(u) \mid^2 \leq \alpha_{ii}(u)\alpha_{jj}(u). \tag{5.10}$$

But, from (5.6), by putting $y = 1$ in formula (5.5) we get for all $1 \leq i \leq n$

$$\alpha_{ii}(u) \leq \Im F_{ii}(u, \sqrt{-1}),$$

and thus

$$\mid \alpha_{ij}(u) \mid \leq \max_{\{u \in K,\ 1 \leq k \leq n\}} \mid \Im F_{kk}(u, \sqrt{-1}) \mid. \tag{5.11}$$

Similarly, from (5.6), for all $1 \leq i, j \leq n$

$$d\mu_{ii}(u,t) \geq 0, \quad \mid d\mu_{ij}(u,t) \mid^2 \leq d\mu_{ii}(u,t) d\mu_{jj}(u,t), \tag{5.12}$$

and thus from (5.12) and the Cauchy-Schwarz inequality it follows,

$$\Big(\int_{-\infty}^{\infty} \frac{\mid d\mu_{ij(u,t)} \mid}{\big((x-t)^2 + y^2\big)^2}\Big)^2 \leq \int_{-\infty}^{\infty} \frac{d\mu_{ii}(u,t)}{\big((x-t)^2 + y^2\big)^2} \int_{-\infty}^{\infty} \frac{d\mu_{jj}(u,t)}{\big((x-t)^2 + y^2\big)^2}.$$

Now, for each of the factors in the right-hand side of the inequality above we have by the inequalities on the left in (5.12) and in (5.10), always for (u, ω) in $K \times U_1$,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{d\mu_{ii}(u,t)}{\big((x-t)^2 + y^2\big)^2} \leq \int_{-\infty}^{\infty} \frac{d\mu_{ii}(u,t)}{(x-t)^2 + 1} &\leq \Im F_{ii}(u, x + \sqrt{-1}) - \alpha_{ii}(u) \\ &\leq \Im F_{ii}(u, x + \sqrt{-1}).\end{aligned}$$

So,

$$\Big(\int_{-\infty}^{\infty} \frac{|d\mu_{ij(u,t)}|}{\big((x-t)^2 + y^2\big)^2}\Big)^2 \leq \Im F_{ii}(u, x + \sqrt{-1}) \Im F_{jj}(u, x + \sqrt{-1}) \tag{5.13}$$

and hence by (5.9), (5.11) and (5.13),

$$|F'_{ij}(u,\omega)| \leq c := 2 \max_{\{u \in K, ;|x| \leq 1\, ;1 \leq k,l \leq n\}} \mid \Im F_{kl}(u, x + \sqrt{-1}) \mid, \tag{5.14}$$

which implies

$$\mid \det F'(u,\omega) \mid \leq n! c^n,$$

showing the inequality (5.3). $\square$

End of Proof of Proposition 3.2. To find the growth estimate for δ throughout $\{\tau \geq 1\}$ (and hence on all of X) from the estimates along the leaves obtained earlier we simply use that the level set $\{\tau = 1\}$ being compact can be covered by a finite number of images of geodesic flow boxes. In fact, it is well-known that given z in X with $\tau(z) = 1$ we may find positive numbers r_1 and r_2 and a real analytic map

$$f \colon \{u = (u_1, \cdots, u_{2n-2}) \in \mathbf{R}^{2n-2} | \sum u_i^2 \leq r_1\} \times [-r_2, r_2] \to \{\tau = 1\}$$

with $f(0,0) = z$ and such that if γ_u denotes the unit speed geodesic with initial conditions $\dot{\gamma}_u(0) = f(u,0)$ and $\gamma_u(0) = \pi(f(u,0))$ then, $\gamma_u(x) = f(u,x)$ for (u,x) in the domain above. (We already used these maps at the end of the proof of Prop. 2.1 but here we are more explicit.) Letting u vary in $\mathbf{R}^{2n-2}$ so that $\sum u_i^2 \leq r_1$ we get a family of geodesics γ_u and leaves $\mathcal{L}_{\gamma_u}$ that depend real analytically on u. Similarly for the vector fields, Jacobi and Fermi, along γ_u, for their extensions to $\mathcal{L}_{\gamma_u}$ and for the parametrizations P_{γ_u}. Performing the construction in section 4, now with u as a real analytic parameter, we obtain a family of matrices $h_{\gamma_u}(\omega)$ of meromorphic functions as in (4.9) depending real analytically on u. We remark that the poles of these matrices as functions in $\mathbf{C}$ lie on the "real axis", hence away from the region where we use them for the estimates. Thus the function

$$F(u,\omega) := h_{\gamma_u}(\omega)$$

satisfies the conditions of Proposition 5.1 (exceedingly so, since only continuity in u would suffice there). The images of finitely many of maps like the f above will cover the level set $\{\tau = 1\}$ and it follows that we can find a pair of constants C_1 and C_2 as in Proposition 5.1 appropiate for all the corresponding functions $F's$. The estimate (3.10), in fact a stronger one,

$$1/\delta \leq c\tau^n, \tag{5.15}$$

now follows from applying Proposition 4.1. □

The Proof of Theorem 1 is at this point complete.

6. Rationality of the holomorphic metric

Since the Riemannian metric g induced by $\sqrt{-1}\partial\bar{\partial}\tau$ along M is real analytic, it extends as a holomorphic object $\tilde{g}$ to a neighborhood $\mathcal{U}$ of M in X, i.e., $\tilde{g}$ is the holomorphic "metric" in $T^{1,0}\mathcal{U}$ continuing the $\mathbf{C}$-bilinear extension to $T^{1,0}X\mid_M$ of the Riemannian metric on M defined by

$$\tilde{g}(\xi^{1,0}, \eta^{1,0}) = g(\xi, \eta)$$

for all ξ and η in TM. Along each leaf of the Monge-Ampère foliation $\tilde{g}$ has the holomorphic Fermi fields as an "orthonormal" basis, i.e., $\tilde{g} = \sum_i^n F_i \otimes F_i$. It follows from an argument similar to the one used in our proof of the extension of the volume form in Proposition 2.1 that $\tilde{g}$ extends to a holomorphic non-degenerate section of the symmetric product of the holomorphic cotangent bundle on the entire tube X.

We will show that such a section is actually rational on the affine $X \subset C^N$. Equivalently, if $\bar{X}$ is a smooth compactification of X, i.e., $X \subset \bar{X}$ and $D := \bar{X} \setminus X$ is a subvariety of dimension less than $n = \dim X$, then $\tilde{g}$ has a meromorphic extension to all of $\bar{X}$ which is holomorphic and non-degenerate on X.

THEOREM 3. *Let (X, τ) be an entire Grauert tube. The τ-induced Riemannian metric on $\{\tau = 0\} = M$ extends as a rational holomorphic metric on all of $\tilde{X}$. Consequently, the holomorphic volume form in Proposition 2.1 extends as a rational form on $\tilde{X}$.*

This will follow from Proposition 6.1 below, whose proof involves our previous estimates and some facts from [**7**]. Especially Theorem 8.5 there, which shows that the transcendence degree over $\mathbf{C}$ of the field of fractions of ϕ-polynomials, K_ϕ, is at most $n = \dim X$, implies that any ϕ-polynomial is a rational function in X.

Let's put $\tilde{g}^*$ for the dual metric on $(T^{1,0}X)^*$. Recall that $\beta = \sqrt{-1}\partial\bar{\partial}(e^\phi)$ with ϕ as in Theorem 1.

PROPOSITION 6.1. *The length of $\tilde{g}^*$ in the metric induced by β satisfies*

$$\parallel \tilde{g}^* \parallel_\beta \leq c_1 e^{c_2\sqrt{\tau}}$$

for constants c_i.

PROOF. Along a parametrized leaf we have, outside the real discrete set S, omitting variables, $F_i = \sum_{k=1}^n C_{ki}\xi_k^{1,0}$, where $C = A^{-1}$. So,

$$\partial\bar{\partial}\tau(F_i, \bar{F}_j) = \sum_{k,l=1}^n C_{ki}\bar{C}_{lj}\partial\bar{\partial}\tau\big(\xi_k^{1,0}, \xi_l^{0,1}\big)$$

and thus, since by (4.25)

$$\sum_{k=1}^n C_{ik}C_{jk} = h'_{ij},$$

we have, by (4.11) and (4.18),

$$\begin{aligned}
\sum_{i,j=1}^n \big(\partial\bar{\partial}\tau(F_i, \bar{F}_j)\big)^2 &= y^2 \sum_{i,j,k,l,s,t=1}^n C_{ki}\bar{C}_{lj}C_{si}\bar{C}_{tj}(\Im h)^{-1}_{kl}(\Im h)^{-1}_{st} \\
&= y^2 \sum_{i,j,k,l,s,t=1}^n C_{ki}C_{si}\bar{C}_{lj}\bar{C}_{tj}(\Im h)^{-1}_{kl}(\Im h)^{-1}_{st} \\
&= y^2 \sum_{k,l,s,t=1}^n h'_{ks}\bar{h}'_{lt}(\Im h)^{-1}_{kl}(\Im h)^{-1}_{st}.
\end{aligned}$$

It follows, since $\Im h$ is positive-definite and hence so is $(\Im h)^{-1}$, that

$$\sum_{i,j=1}^n \big(\partial\bar{\partial}\tau(F_i, \bar{F}_j)\big)^2 \leq y^2 \sum_{k,l,s,t=1}^n \mid h'_{ks} \mid\mid \bar{h}'_{lt} \mid \big((\Im h)_{kk}(\Im h)_{ll}(\Im h)_{ss}(\Im h)_{tt}\big)^{-\frac{1}{2}}.$$

Now we use Proposition 5.1 with $F = h(u, \omega)$ to estimate the right-hand side above. From (5.7) it follows that

$$(\Im F)^{-1}(u, \omega) \leq 4y(\Im F)^{-1}(u, \sqrt{-1})$$

(for (u, ω) in an appropriate $D_{r_1} \times D_{r_2}$) and this together with the estimate (5.14) implies, after arguing as in the end of the proof of Proposition 3.2, that for $\tau \geq 1$ there is a constant $c > 0$ so that

$$\mid \partial\bar{\partial}\tau\big(F_i, \bar{F}_j\big) \mid \leq c\tau, \tag{6.1}$$

where the inequality (6.1) with the same constant c is valid for any set of holomorphic Fermi fields along any parametrized geodesic.

Using (6.1) and the expression for the β-metric (3.5) it is now easy to verify that there is a constant $c_1 > 0$ so that

$$\|F_i\|_\beta^2 \leq c_1 e^{2\sqrt{\tau}}. \tag{6.2}$$

(For this calculation one recalls the splitting induced by the kernel of $\partial\tau$ and the evaluation of the F_k on $\partial\tau$ and $\partial\bar{\partial}\tau$ as discussed in the proof of Prop. 2.1.) Now, from 11.5(c) in [**7**], and with $C_1 \gg 0$, $\rho \equiv 0$, for any point $z \in X$, we can find $f_1, \cdots, f_n$, such that

$$f_i \in \mathrm{A}^b_\phi$$

(and therefore "ϕ-polynomials" by Lemma 11.3 in [**7**]), where

$$\mathrm{A}^b_\phi = \{f\colon X \to \mathbf{C}\ \textit{holomorphic} \ \mid\ \exists C \geq 0\ \mid\ \int_X \mid f\mid^b e^{-C\phi}\beta^n < \infty n\},$$

satisfying

$$df_1 \wedge df_2 \cdots \wedge df_n \neq 0,$$

at $z \in X$. Here $b = \frac{2c}{1+c} \in (0, 2)$. Thus, the differentials df_i give a rational frame for the holomorphic cotangent bundle $T^*_{(1,0)}(X)$ on a Zariski open set of X. The coefficients of the dual holomorphic metric $\tilde{g}^*$ on $T^*_{(1,0)}(X)$ are given, in this frame and on this Zariski open set, by

$$\tilde{g}^*(df_i, df_j),$$

and we have succeeded in showing that g is rational if these coefficients are "ϕ-polynomials" as indicated earlier. In fact, we can show that they are in $\mathrm{A}^{\frac{b}{2}}_\phi$. To see this latter, consider

$$\int_X \|\tilde{g}^*(df_i, df_j)\|_\beta^{\frac{b}{2}} e^{-\tilde{C}\phi}\beta^n.$$

The integrand can be estimated:

$$\|\tilde{g}^*(df_i, df_j)\|_\beta^{\frac{b}{2}} \leq \|\tilde{g}^*\|_\beta^{\frac{b}{2}}\ \|df_i\|_\beta^{\frac{b}{2}}\ \|df_j\|_\beta^{\frac{b}{2}}.$$

But by (6.2), we can estimate

$$\|\tilde{g}^*\|^{\frac{b}{2}} \leq c_1' e^{c_2'\sqrt{\tau}}$$

which can be absorbed into the weight term if $\tilde{C} \gg 0$. Again, perhaps taking $\tilde{C}$ larger, we can then use the Cauchy-Schwarz inequality to conclude that the coefficients are in $\mathrm{A}^{\frac{b}{2}}_{\phi}$. This last step uses 11.3 (b) in [**7**]. This shows Proposition 6.1 and, as indicated above, in virtue of Theorem 8.5 in [**7**], Theorem 3 is thus proved. $\square$

The rationality of $\tilde{g}$ is of interest in connection with the problem of characterizing those Riemannian manifolds that appear as "centers" of entire Grauert tubes, a question we hope to come back to in the future. But for now, we will derive from Theorem 3 some natural corollaries related to that problem.

In general, an entire Grauert tube (X, τ_1) is non-rigid in the sense that it might support another Monge-Ampère exhaustion τ_2 different from a constant multiple of τ_1 with the same zero set M. Thus the Riemannian structures induced in M by the different Monge-Ampère functions are non-isometric, even non-homothetic. (Note that such τ_2 is necessarily unbounded. This is so because the maximum on the level set $\{\tau_1 = r\}$ of any plurisubharmonic function such as τ_2 is increasing convex as a function of r. See Corollary 6.6 in [**7**].)

The rationality theorem above yields, under an additional assumption on $D = \bar{X} \setminus X$, where $\bar{X}$ is a smooth, projective compactification of the affine manifold X, the "rigidity" of the Riemannian volume in M as follows.

Corollary 6.1.1. *Let τ_i be two Monge-Ampère exhaustions on X so that (X, τ_i) is an entire Grauert tube with $M = \{\tau_1 = 0\} = \{\tau_2 = 0\}$. If the divisor $D = \tilde{X} \setminus X$ is irreducible then the Riemannian volume forms on M corresponding to the Riemannian metrics g_i induced by τ_i are, up to homothety, identical.*

Proof. The argument in the previous proof shows that the holomorphic forms $\mathcal{V}_i$ extending the Riemannian volume forms of (M, g_i) extend in turn as meromorphic sections $\mathfrak{V}_i$ of the canonical bundle $T^{n,0}(\tilde{X})$ of the compactification $\tilde{X}$. By Proposition (2.1) $\mathfrak{V}_i$ are non-vanishing on X. Thus

$$\mathfrak{V}_1 = f\,\mathfrak{V}_2$$

where f is a meromorphic function on $\tilde{X}$, holomorphic and non-vanishing on X. Assume f is not constant. Then if D is the polar set of f (or of $1/f$) by the Riemann Extension Theorem $1/f$ (or f) extends as a non-constant holomorphic function on the compact $\tilde{X}$, which is not possible. $\square$

Examples of unbounded Grauert tubes with $D = \tilde{X} \setminus X$ an irreducible hypersurface are the standard complexifications $G_{\mathbf{C}}/H_{\mathbf{C}}$ of compact symmetric spaces of rank one, $M = G/H$, G a compact connected semi-simple Lie group and $H \subset G$ a closed subgroup. The Stein manifold $G_{\mathbf{C}}/H_{\mathbf{C}}$ can be real-analytically identified with the cotangent bundle of G/H, $T^*(G/H)$, in a G-equivariant way and so that $G/H \subset G_{\mathbf{C}}/H_{\mathbf{C}}$ is identified with the zero section. The Monge-Ampère exhaustion τ is the function on $G_{\mathbf{C}}/H_{\mathbf{C}}$

corresponding to the length squared function on $T^*(G/H)$ induced by the metric on G/H. From the explicit realization of $G_{\mathbf{C}}/H_{\mathbf{C}}$ as affine manifolds (cf. [**14**]) we see that the hypothesis on the divisor at infinity for their compactification is satisfied.

Finally, note that the anti-holomorphic involution σ operates on holomorphic functions on X by $\sigma(f)(z) = \overline{f(\sigma(z))}$. This action leaves invariant the various ϕ-polynomial norms. Given the affine embedding of X by ϕ-polynomials, $(f_1, ..., f_N)$, we may expand to the affine embedding with components $(f_1, ..., f_n, \sigma(f_1), ..., \sigma(f_N))$, and after composing with a linear change of variable in $\mathbf{C}^{2N}$, we may arrange that we have an affine embedding of X into $\mathbf{C}^{2N}$ which is σ-equivariant, that is:

$$(F_1(\sigma(z)), ..., F_{2N}(\sigma(z))) = (\overline{F_1(z)}, ..., \overline{F_{2N}(z)}).$$

This will mean that the projective closure $\overline{F(X)}$ of $F(X) \subset \mathbf{C}^{2N}$ inside $\mathbf{P}^{2N}$ will be a real variety invariant under complex conjugation. By [**3**], there is a smooth resolution $\widehat{F(X)}$ of $\tilde{X}$ with an exceptional divisor with normal crossings which is invariant under the lift of σ to $\hat{\sigma} : \widehat{F(X)} \to \widehat{F(X)}$. It is also easy to check that the rational metric $\tilde{g}$, for example, is σ-invariant on X in the natural sense.

7. Proof of Theorem 2

We continue identifying the entire Grauert tube (X, τ) with the tangent bundle TM of $M = \{\tau = 0\}$, endowed with the adapted complex structure $\mathbf{J}$. According to Lempert and Szőke (Theorem 4.1 in [**12**]) the metric g on M has non-negative Gauss curvature. By Gauss-Bonnet, the Euler characteristic of M is non-negative, and so the center of the tube is *isometric* to a flat two-torus T^2 or or flat Klein bottle [**12**], or diffeomorphic to a two-sphere S^2 or real projective plane.

By the uniqueness of the adapted complex structure, in the case of the flat two-torus the Grauert tube is biholomorphic to $\mathbf{C^2}/\Lambda \simeq \mathbf{C}^* \times \mathbf{C}^*$, where Λ is a lattice in $\mathbf{R^2}$, or a $\mathbf{Z_2}$ quotient of it, in the case of the Klein bottle.

Again by functoriality and uniqueness of the adapted complex structure the projective plane case is subsumed into the one which we now treat, M diffeomorphic to S^2.

By Theorem 1 the Grauert tube X can be imbedded algebraically in some $\mathbf{C}^N$ so that X is a Zariski open set of a smooth projective compact variety $\widetilde{X}$. Thus, $\tilde{X}$ is a smooth projective compactification of X. The "divisor at ∞" is given by,

$$\widetilde{X} \setminus X = D = \bigcup_{i \in I} D_i$$

where D_i are Riemann surfaces which, by blowing-up if necessary, will be assumed to intersect normally, i.e., *if* $i \neq j$, $D_i \cap D_j = \{$ *point* $\}$ *or* $= \emptyset$ and *if* $i \neq j \neq k$ $D_i \cap D_j \cap D_k = \emptyset$. Here each D_i is an irreducible

component obtained as the closure of a connected component of the regular part of D.

We now argue as Ramanujam in [**15**]. In what follows homology and cohomolgy is taken with rational coefficients $\mathbf{Q}$, unless indicated differently. Let N be a tubular neighborhood of $D \subset \tilde{X}$, a four-dimensional manifold with boundary ∂N. For $r > 0$,

$$\partial N \simeq \{\tau = r\} \simeq \mathbf{SO}(3, \mathbf{R}) \simeq S^3/\mathbf{Z}_2.$$

Thus ∂N is connected and so is D, a deformation retract of N. Moreover $H^1(\partial N) \simeq H_2(\partial N) = 0$. On the other hand

$$H^k(N, \partial N) \simeq H_{4-k}(N) \simeq H_{4-k}(D)$$

for $0 \leq k \leq 4$, the first identification by the relative Poincarè duality. But, since $H_3(D) = 0$, $H^1(N, \partial N) = 0$, and hence it follows from the exact sequence $\ldots \mapsto H^1(N, \partial N) \mapsto H^1(N) \mapsto H^1(\partial N) \mapsto \ldots$, that $H^1(N) = 0$ and so $H^1(D) = 0$. As a consequence, each $D_i \simeq \mathbf{P^1}$, the complex projective plane, and moreover there are no “loops”, i.e., there is no sequence of distinct values $\{i_1, \ldots, i_N\}$ for which $D_{i_1} \cap D_{i_N} \neq \emptyset$ and $D_{i_k} \cap D_{i_{k+1}} \neq \emptyset,$ *for all* $1 \leq k \leq N-1$.

The *intersection matrix* for D has coefficients (I_{ij}) defined by: I_{ii} = self-intersection number of D_i, $I_{ij} = 1$ if $D_i \cap D_j \neq \emptyset$ and $i \neq j$ and $I_{ij} = 0$ otherwise. From the discussion above, to the data $\{D_i\}$ there is associated a weighted tree T with a vertex v_i for each curve D_i with the weight on v_i equal to I_{ii}, two distinct vertices v_i and v_j being linked if and only if D_i and D_j intersect. It is well-known that the fundamental group of $\pi(\partial N)$ is isomorphic to the “fundamental group” $\pi(T)$ of the tree T, a presentation of which is given by the set of vertices $\{v_i\}$ subject to the set of relations $\{R_i, i \in I\}$

$$\{R_i\} = \begin{cases} v_i v_j^{I_{ij}} = v_j^{I_{ij}} v_i & 1 \leq i, j \leq N \\ 1 = v_1^{I_{i1}} \cdots v_N^{I_{iN}} & 1 \leq i \leq N \end{cases} \tag{7.1}$$

where in the second line in the relations $\{R_i\}$ the product of the $v_j^{I_{ij}}$ a fixed order (irrelevant up to isomorphism) of all the vertices is assumed.

A classical theorem of Mumford [**13**] asserts that if the fundamental group of the connected tree associated to a connected divisor with negative-definite intersection matrix is trivial, then the divisor can be blown-down to a smooth point. We will later refer to this result as “Mumford Theorem”. In the next propositions we determine how to choose subtrees of T to which we can apply Mumford’s theorem.

As explained after the statement of Corollary 6.1.1, we can assume the resolution $\tilde{X}$ to be so that the canonical anti-holomorphic involution $\sigma \colon X \to X$ extends as an anti-holomorphic involution $\tilde{\sigma} \colon \tilde{X} \to \tilde{X}$, which then acts on D. Since $\tilde{X}$ is a complex compact two-dimensional Kähler manifold, the cup product restricted to $H^{1,1}(\tilde{X}) \cap H^2(\tilde{X}, \mathbf{Z})$ has exactly a 1 dimensional subspace where it is positive definite, by the Hodge index theorem [**8**]. We

will next show that such a subspace is, after bowing down, the span of one of the D_i's. So the strategy now is first to identify the part of $D = \tilde{X} \setminus X$ that is invariant by $\tilde{\sigma}$, which is done in Proposition 7.1 and then to show that its transform after blowing down the complement, has positive self-intersection, which is proved in Proposition 7.3. The final step will be to show that the surface is actually rational.

PROPOSITION 7.1. *We may assume $D = D_0 \cup_{j>0} \{D_j \cup D_{-j}\}$ with $D_j \neq D_k$ for $j \neq k$, $\tilde{\sigma} D_0 = D_0$ and $\tilde{\sigma} D_{\pm j} = D_{\mp j}$, with $\tilde{\sigma}|_{D_0} \neq$ Identity .*

PROOF. Since the compactification is assumed invariant with respect to $\tilde{\sigma}$, we have that $\tilde{\sigma} D = D$. To find out how many components D_i are non-trivially permuted by $\tilde{\sigma}$, i.e., $\tilde{\sigma} D_i \neq D_i$, we consider the action of $\tilde{\sigma}$ on $H^*(\tilde{X})$.

Because D has the homotopy type of a wedge product of N 2-spheres, $H^2(D) \simeq \mathbf{Q}^N$. But,

$$H^k(\tilde{X}, D) \simeq H^k_{cpt.}(X) \simeq H_{4-k}(X) \simeq H_{4-k}(S^2),$$

where $H_{cpt.}$ denotes cohomology with compact support, and by the exact sequence $\ldots \mapsto H^k(\tilde{X}, D) \mapsto H^k(\tilde{X}) \mapsto H^k(D) \mapsto \ldots$ we get

$$H^2(\tilde{X}) \simeq \mathbf{Q}^{N+1}, \quad H^k(\tilde{\mathcal{X}}) \simeq 0, \quad k = 1 \textit{ or } k = 3. \tag{7.2}$$

Since $\tilde{X}$ is connected, orientable, and the complex dimension of $\tilde{X}$ is even, $\tilde{\sigma}_*$ is the identity on $H^0(\tilde{\mathcal{X}}) \simeq \mathbf{Q}$ and also on $H^4(\tilde{X}) \simeq \mathbf{Q}$. Now, $\tilde{\sigma}$ restricted to $\{\tau = 0\}$ is the identity map, and this gives an additional 1-dimensional subspace of $H^2(\tilde{X})$ where $\tilde{\sigma}_* = I$. But each connected component of the fixed-point set of the anti-holomorphic involution $\tilde{\sigma}$ is a totally real submanifold of $\tilde{X}$. It follows that if $\tilde{\sigma} D_i = D_i$ then $\tilde{\sigma}$ can not be the identity on such D_i but must induce an antiholomorphic involution. In particular $\tilde{\sigma}$ restricted to any invariant D_i is orientation reversing and induces $-$Identity on the subspace in $H^2(\tilde{X})$ spanned by such D_i.

So, from the discussion above, letting ν be the number of the D_i's invariant by $\tilde{\gamma}$, by the Lefschetz Fixed Point Theorem

$$\sum_{k=0}^{4} (-1)^k \textit{trace } (\tilde{\sigma}_*|_{H^k(\tilde{X})}) = 1 - 0 + (1 - \nu) - 0 + 1 = \epsilon(S^2) = 2,$$

for the total contribution to the trace of those $N - \nu$ subspaces of $H^2(\tilde{X})$ corresponding to the D_i that are non-trivially permuted is zero.

Thus, $\nu = 1$, i.e., there is only one curve of D left invariant by $\tilde{\sigma}$ which we will denote by D_0 and call the "central curve". $\square$

PROPOSITION 7.2. *Let $\tilde{X}$ and D as above, with N the number of irreducible components of D. Then the Euler and holomorphic Euler characteristics of $\tilde{X}$ are*

$$\epsilon(\tilde{X}) = 3 + N, \quad \chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \sum_{p=o}^{2} (-1)^p h^{0,p} = 1.$$

PROOF. For the Euler characteristic $\epsilon(\tilde{X})$ use (7.2) and for the holomorphic Euler characteristic, the compactness of $\tilde{X}$ and Hodge decomposition. Here the simple connectedness of $\tilde{X}$ implies $h^{01} = \frac{1}{2}h^1 = 0$. Also the subspace of $H^{1,1}(\tilde{X})$ generated by D has codimension 1 in $H^2(\tilde{X})$ and thus $h^{0,2} = 0$. □

PROPOSITION 7.3. *The set of curves $\bigcup_{i\neq 0} D_i$ can be blown-down.*

PROOF. Let $F_{\pm k}$ be the connected components of

$$\bigcup_{i\neq 0} D_i = \bigcup_{k>0}^{N'} F_k \cup F_{-k},$$

where $2N'$ is the number of D_i's meeting the central curve. Here $\tilde{\sigma} F_k = F_{-k}$, and so, F_k being orthogonal to F_{-k} and the intersection product invariant by $\tilde{\sigma}$, by the Hodge-Riemann index theorem, the intersection product on each F_k must be negative definite. Thus, by Mumford Theorem, a connected component F_k can be blown-down provided that the fundamental group of the branch T_k on T associated to F_k satisfies

$$\pi(T_k) = 1. \tag{7.3}$$

Now, consider the group $\pi(T)/(v_0)$ which may be presented by adding the relation $v_0 = 1$ in (7.1) or, equivalently, by the set of generators $\{v_1, \cdots, v_N\}$ with the relations

$$\tilde{R}_i = \begin{cases} \{R_i\} \textit{ for } i \neq 0 \\ 1 = v_{j_1} \cdots v_{j_{2N'}} \end{cases}$$

where in the last relation the generators involved are those corresponding to the vertices linked to v_0 in the original tree T. Except this last relation, there are no further relations among generators corresponding to vertices in different branches T_k's. Thus, this group may be identified with the amalgamated free product

$$(\pi(T_{-N'}) * \cdots * \pi(T_{-1})) * \pi(T_1)) * \cdots * \pi(T_{N'}))/(v_{j_1} * \cdots * v_{j_{2N'}}). \tag{7.4}$$

We also need the following

PROPOSITION 7.4. [**13**] *Let G_i, $i = 1, 2, 3$ be any non-trivial groups, and let $a_i \in G_i$ be arbitrary elements. Then the amalgamated free product*

$$\big(G_1 * G_2 * G_3\big)/\big(a_1 * a_2 * a_3\big)$$

is non-trivial non-cyclic.

Since $\pi(T)/(v_0)$ is either trivial or $\mathbf{Z}_2$ from (7.4), Proposition 7.4 and Mumford Theorem applied inductively, we may assume that, after blowning down pairs of branches $\{F_{-k}, F_k\}$ in D, there are at most two branches F_- and F_+ say, interchanged by the involution $\tilde{\sigma}$ with $\pi(T_+) \simeq \pi(T_-)$. In particular, we have

$$\pi(T)/(v_0) \simeq \big(\pi(T_+) * \pi(T_-)\big)/\big(v_+ * v_-\big).$$

Claim: $\pi(T_+) \simeq \pi(T_-) \simeq 1$.

Proof of Claim. Assume T_+ has K vertices. In an appropiate basis we write down the intersection matrix restricted to the span of the divisor D as the matrix with integer coefficients,

$$\mathfrak{I} = \begin{pmatrix} a & C & C \\ C^T & B & 0 \\ C^T & 0 & B \end{pmatrix} \in \mathbf{Z}^{(K+1)^2}$$

where, the superscript $''T''$ as earlier indicates transpose, each block B is a symmetric matrix in $\mathbf{Z}^{K\times K}$ corresponding to the K vertices in T_+ or to the K vertices in T_-, C is a $1\times K$ matrix with one entry equal to 1 and all the other ones equal to 0, and finally $a = I_{00}$ is the weight of the central vertex v_0. Since the intersection form is positive-definite in the span of the central curve and negative-definite in the branches F_+ and F_-, we have $a > 0$, and rank $B = K$.

To prove the Claim all we need to show is that $a > 1$. Indeed, if $a > 1$, then take the groups $G_+ = \pi(T_+)$, $G_- = \pi(T_-)$ and $< v_0 >$, the cyclic group of order a generated by v_0. Let $v_\pm$ the vertex in $T_\pm$ linked to v_0. Then, arguing as in the discussion preceding expression (7.4) we see that

$$\mathbf{Z}_2 \simeq \pi(T) = \big(\pi(T_+) * \pi(T_-) * < v_0 > \big)/\big(v_+ * v_- * v_0\big) \tag{7.5}$$

which, unless $\pi(T_\pm) \simeq 1$, would contradict Proposition 7.4.

So, assuming $a = 1$, we will get to a contradiction. Consider the matrices

$$U_1 = \begin{pmatrix} 1 & -C & -C \\ C^T & I & 0 \\ C^T & 0 & I \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 0 & I & I \end{pmatrix}, \quad U_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & -I \\ 0 & 0 & I \end{pmatrix}$$

$\in \mathbf{Z}^{(K+1)^2}$ with $\det U_i = 1$ for $i = 1, 2, 3$. Then,

$$U_3 U_1^T \mathfrak{I} U_1 U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & B - 2\, C^T C & -C^T C \\ 0 & 0 & B \end{pmatrix}$$

Thus, since $\pi(T) = \mathbf{Z}_2$,

$$2 = |\det \mathfrak{I}| = |\det B||\det(B - 2\, C^T C)|.$$

On the other hand, since we are assuming $a = 1$, it follows from (7.5) that $|\det B| > 1$. Also, $B - 2C^TC$ has integer coefficients, so we must have

$$|\det B| = 2, \quad |\det(B - 2\,C^TC)| = 1. \tag{7.6}$$

But if $|\det B| = 2$ then $2B^{-1}$ is well defined in $\mathbf{Z}^{(K+1)^2}$, $I - 2B^{-1}C^TC$ has integer coefficients and hence,

$$\det(I - 2B^{-1}C^TC) \in \mathbf{Z}. \tag{7.7}$$

Then, by the right handside equality in (7.6)

$$1 = |\det(B - 2\,C^TC)| = |\det B||\det(I - 2\,B^{-1}C^TC)|,$$

a contradiction in virtue of (7.7) and the left handside equality in (7.6).

We showed $a > 1$ and $\#\pi(T_\pm) = |\det B| = 1$, thus proving the Claim. This finishes the proof of Proposition 7.3. □

7.0.2. *Rationality of the surface.* An algebraic surface is called rational if it is birational to the complex projective plane $\mathbf{P}^2$, which is equivalent to saying that it may be obtained from it by blowing up points followed by blowing down curves.

Because of the results in the previous sections , the divisor at infinity in the algebraic surface $\tilde{X}$, $\tilde{X} \setminus X$, may be assumed to consist of just a $\mathbf{P^1}$ with self-intersection $a = 2$. This together with the topology of X, we will show, implies that $\tilde{X}$ verifies the conditions of the classical Castelnuovo Criterion for Rationality (cf. [**8**]), namely zero first Betti number and zero geometric genus P_2 (7.9). Moreover, the topology of $\tilde{X}$ will force it to be "minimal", i.e. to contain no curve with self-intersection -1, and thus to belong to a certain family of rational surfaces $\mathbf{F}_k$ which are projective bundles over $\mathbf{P}^1$. From here it will follow that $\tilde{X}$ is $F_0 = \mathbf{P}^1 \times \mathbf{P}^1$.

PROPOSITION 7.5. *$\tilde{X}$ is a minimal rational surface.*

PROOF. Rationality first. Let $\tilde{X}$ be the compactification of X after blowing down all the curves in $\bigcup_{i\neq 0} D_i$, and still denote by D the image of D_0 after the blowing down process. Let K denote the canonical divisor class of $\tilde{X}$. Since D is a smooth rational curve we have, from the genus formula (cf. [**8**])

$$D \cdot (D + K) = 2g(D) - 2 = -2$$

and, since $D \cdot D = 2$, it follows that

$$D \cdot K = -4. \tag{7.8}$$

This implies that for all $m > 1$ the line bundle $\otimes^m K$ have no holomorphic sections, because any such would define an effective divisor which would have a non-negative intersection with D, contradicting (7.8). In particular,

$$P_2 := \ dim\ H^0(\tilde{X}, \otimes^2 K) = 0. \tag{7.9}$$

Since the first Betti number of $\tilde{X}$ is zero as well, the rationality of $\tilde{X}$ follows from the rationality criterion of Castelnuovo.

We now show that $\tilde{X}$ satisfies the "minimality condition". This can be determined from the "even parity" of the intersection form. Indeed, according to a well-known theorem of H. Wu, since $\tilde{X}$ is simply connected, the stronger requirement that the square of any element in $H^2(\tilde{X}, \mathbf{Z})$ be even, is equivalent to $\tilde{X}$ being spin, i.e., its second Stiefel-Whitney class

$$\omega_2(\tilde{X}) = c_1(\tilde{X}) \mod (2) = 0. \tag{7.10}$$

To show that indeed (7.10) holds in our case, recall that $c_1(\tilde{X}) = -c_1(K)$. Moreover, $H^2(\tilde{X})$ is generated over $\mathbf{Q}$ by D and the class S corresponding to $\tau^{-1}(0)$ and we may write

$$K = aD + dS, \quad a, b \in \mathbf{Q}.$$

Note also that $S \cdot S = -2$ (recall, the Euler number of the unit sphere bundle of the tangent bundle of $\tau^{-1}(0)$).

On the other hand, from Proposition 7.2 and Noether's Theorem (cf. [**8**]) which reads

$$1 = \chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \frac{1}{12}(K \cdot K + \epsilon(\tilde{X})),$$

we get $K \cdot K = 8$, hence $2a^2 - 2b^2 = 8$. But, since $\tau^{-1}(0)$ and the divisor at ∞ are disjoint, and hence their classes orthogonal, using (7.8),

$$-4 = D \cdot K = D \cdot (aD + bS) = 2a.$$

Thus, we have $b = 0$ and

$$K = -2D, \tag{7.11}$$

from which (7.10) follows, thus the minimality of $\tilde{X}$. $\square$

Proposition 7.6. $\tilde{X} \simeq \mathbf{P}^1 \times \mathbf{P}^1$.

Proof. It is a classical result in the theory of algebraic surfaces that a minimal rational surface, such as $\tilde{X}$, is either $\mathbf{P}^2$ or belongs to a certain family of $\mathbf{P}^1$ bundles $\mathbf{F}_k$ over $\mathbf{P}^1$, sometimes referred to as Hirzebruch surfaces (cf. [**8**]). Clearly, the first possibility is ruled out because $\tilde{X} \setminus D$ is homotopically equivalent to S^2, in particular, non-contractible, and thus we are left with the second possibility.

Each $\mathbf{F}_k$, $0 \leq k$, is the projectivization of a rank 2 bundle, namely,

$$\mathbf{F}_k = \mathbf{P}_{\mathbf{P}^1}(\mathcal{O}_{\mathbf{P}^1} \oplus (\otimes^k \mathcal{O}_{\mathbf{P}^1}))$$

where $\mathcal{O}_{\mathbf{P}^1}$ is the degree 1 bundle over $\mathbf{P}^1$. Given such a surface, the value of k is determined by the self-intersection of the curve C which is the image of the section defined by the subbundle

$$\mathcal{O}_{\mathbf{P}^1} \oplus \{0\} \subset \mathcal{O}_{\mathbf{P}^1} \oplus (\otimes^k \mathcal{O}_{\mathbf{P}^1})).$$

All we need is the fact that for any such surface,

$$C \cdot C \leq 0, \tag{7.12}$$

with equality if and only if the surface is $\mathbf{F}_0 \simeq \mathbf{P}^1 \times \mathbf{P}^1$.

But, in our case we must have

$$C \cdot D \geq 1 \tag{7.13}$$

since otherwise $C \cdot D = 0$ and C would be contained in the affine part $X = \tilde{X} \setminus D$, which is impossible. It follows using (7.12), (7.13) and (7.11) in the genus formula for C that

$$0 \leq C \cdot D - 1 = \frac{1}{2} C \cdot C \leq 0,$$

and thus $C \cdot C = 0$, which implies

$$\tilde{X} \simeq F_0 \simeq \mathbf{P}^1 \times \mathbf{P}^1.$$

It follows that the σ is given by $\sigma(z, w) = (\bar{w}, \bar{z})$, in suitable coordinates, since otherwise the fixed points of σ would be homeomorphic to the torus of dimension two. The curve D at infinity must be homologous to $C_1 + C_2$, where the C_i are the two factors in $\tilde{X}$. It follows that $\tilde{X}$ is embedded as the usual quadric in $\mathbb{P}^3$, and that σ is conjugate linear in the homogeneous coordinates of $\mathbb{P}^3$, with fixed point set homeomorphic to S^2. Without loss of generality, we can conjugate $\tilde{X}$ by an element $g \in \mathrm{Aut}_{\mathbb{C}}(\tilde{X})$ so that D is the standard diagonal in $\tilde{X}$, the curve at infinity for the Grauert tube for the round metric on S^2. The subgroup G_D of $\mathrm{Aut}_{\mathbb{C}}(\tilde{X})$ sending this D into itself is isomorphic to the adjoint group of $SO(3, \mathbb{C})$, and G_D has two real forms corresponding to σ and the standard conjugation σ_0 of $\tilde{X}$ coming from the round metric. The topology of the two sets of real points in $\tilde{X}$ shows that both of the real forms of G_D are isomorphic to the real adjoint group of $SO(3, 1)$. Hence, there is an element $g' \in G_D$ conjugating σ to σ_0. Hence, X is biholomorphic to the standard affine quadric, and in such a way that the complex conjugation σ is equal to the standard conjugation σ_0.

$\square$

8. Open Questions

We have shown that Riemannian metrics on compact manifolds M which give rise to entire Grauert tubes X must necessarily be algebraic, and the Riemannian metric must extend to a rational metric on $\tilde{X}$, regular and non-degenerate on X. In the previous section we showed that this suffices in the lowest dimensional case to rigidify the complex structure on X for $M = S^2$. It remains to see why examples of such metrics are so rare. It is a theorem of Szőke's [**17**] that among the surfaces of revolution on S^2 with respect to a given rotation, the only ones corresponding to unbounded tubes form a two-parameter family, where one of the parameters is simply rescaling the metric. (See [**2**] for all known higher dimensional examples.) There are still quite a few rational metrics of the sort given by theorem 1 and its refinements, in part because the tangent bundle of our affine variety X has a very large algebraic gauge group. It is very difficult to encode the real information of τ and the associated Kähler metric on X in the metric

$\tilde{g}$ on X. It would be interesting even to see what the possibilities are on the two sphere S^2.

It is not clear that there is enough known about minimal models in three dimensions to be able to prove the analogue of theorem 2 for metrics on the sphere S^3.

Finally, it is an interesting question whether Demailly's method [**7**] can be used to resolve the algebraicization conjecture of [**4**]. The current paper settles this when X is a Grauert tube.

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